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Quantum theory of optical bistability. I: Nonlinear polarisability model

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Abstract. A quantum treatment of a coherently driven dispersive cavity is given based on a cubic nonlinearity in the polarisability of the internal medium. This system displays bistability and hysteresis in the semiclassical solutions. Quantum fluctuations are included via a Fokker–Planck equation in a generalised P representation. The transmitted light shows a transition from a single-peaked spectrum to a double-peaked spectrum above the threshold of the lower branch. Fluctuations in the field are reduced on the upper branch and both photon bunching and photon antibunching are predicted, for different operating points. An exact solution obtained for the steady-state generalised P function shows decidedly non-equilibrium behaviour, e.g. the lack of a Maxwell construction.

1. Introduction

The possibility of achieving a bistable optical device using a saturable absorber inside a Fabry–Perot cavity was first suggested by Seidel (1969) and Szoke et al (1969). The motivation for constructing such a device is for potential use as a switching element in an optical communications system. The first experimental demonstration of a bistable device was achieved by Gibbs *et al* (1976), who used a nonlinear dispersive medium inside a Fabry–Perot cavity. Both the experiments of Gibbs *et al* (1976) and, more recently, those of Sandle and Gallagher (1978, private communication) have shown that threshold powers are lower in practical dispersive devices, relative to absorptive bistability.

A considerable amount of theoretical work on optical bistability has been performed. The first semiclassical analysis of both dispersive and absorptive bistability was given by McCall (1974). A semiclassical analysis using Maxwell–Bloch equations for absorptive bistability was given by Bonifacio and Lugiato (1976). A semiclassical analysis of a nonlinear dispersive Fabry–Perot interferometer using a nonlinear polaris ability model was given by Marburger and Felber (1978). A full semiclassical analysis of optical bistability in a Fabry–Perot cavity including inhomogeneous broadening has been given by Hassan *et al* (1978) and Bonifacio and Lugiato (1978c). Related analyses, but without inhomogeneous broadening, have been given by Meystre (1978), Agrawal and Carmichael (1979) and Schwendimann (1979). Quantum treatments have in the main been restricted to the absorptive case (Bonifacio and Lugiato 1978a, b, Bonifacio *et al* 1978, Carmichael and Walls 1977, Hassan and Walls 1978, Walls *et al* 1979, Willis 1977, 1978, Narducci *et al* 1978, Agarwal *et al* 1978a, b, Lugiato 1979). An exception is a recent paper by Willis and Day (1979), who have derived a master equation for dispersive bistability.

It is the aim of the present paper to give a full quantum-mechanical treatment of dispersive optical bistability. We shall adopt an approach of considering a cubic nonlinearity in the polarisation, as did Marburger and Felber (1978). We consider a single cavity field mode which is quantised and driven externally by a coherent driving field. Dissipation of the cavity mode is included. The full quantum-mechanical Hamiltonian for this system is constructed. As a first approximation we neglect quantum fluctuations and derive the semiclassical equations of motion. Then quantum fluctuations are included via the use of a generalised P function. This leads to the stochastic differential equations obeyed by the complex field amplitude. Adopting an expansion to first order in the variance of the fluctuations enables expressions for the mean photon number, the second-order correlation function and the spectrum of the transmitted light to be calculated.

In the steady state in the regime of pure quantum fluctuations it is possible to solve exactly for the quasi-probability distribution function in the generalised P representation. This enables exact expressions to be obtained for the mean photon number and the second-order correlation function.

Although in this paper we derive the equations of motion from a phenomenological or macroscopic model for the nonlinear polarisability, we show in a following paper that identical equations are reproduced by a microscopic model for the medium consisting of two level atoms, for large detuning relative to the line width.

2. Hamiltonian and mean field equations

We wish to consider a single-mode field inside a cavity which contains a nonlinear dispersive medium. The single cavity mode is driven externally by a coherent driving field.

The Hamiltonian for the interaction of a single-mode field and a nonlinear dispersive medium may be obtained by expanding the polarisation of the medium to third order in the electric field amplitude:

$$\boldsymbol{P}=\chi^{(1)}\cdot\boldsymbol{E}+\chi^{(2)}:\boldsymbol{EE}+\chi^{(3)}:\boldsymbol{EEE} \tag{2.1}$$

where $\chi^{(n)}$ is a $(n+1)$th rank susceptibility tensor. This yields for the Hamiltonian (Bloembergen 1965)

$$H=:\int \mathrm{d}^3\boldsymbol{r}\{|\boldsymbol{B}|^2/2\mu_0)+\boldsymbol{E}[\tfrac{1}{2}(\epsilon_0+\chi^{(1)})\boldsymbol{E}+\tfrac{1}{3}\chi^{(2)}\boldsymbol{EE}+\tfrac{1}{4}\chi^{(3)}\boldsymbol{EEE}]\}: \tag{2.2}$$

where : : denotes normal ordering. If second harmonic generation can be neglected due to lack of phase matching we may neglect the term in $\chi^{(2)}$.

We make a normal-mode expansion for a single-mode electric field

$$\boldsymbol{E}=\mathrm{i}\left(\frac{\hbar\omega}{2\epsilon_0}\right)^{1/2}(a\boldsymbol{u}(r)-a^{\dagger}\boldsymbol{u}^*(r)) \tag{2.3}$$

where the mode function $\boldsymbol{u}$ is defined to satisfy

$$\int[\boldsymbol{u}^*(r)(1+\chi^{(1)}(\boldsymbol{r})/\epsilon_0)\boldsymbol{u}(r)]\,\mathrm{d}^3\boldsymbol{r}=1. \tag{2.4}$$

The single-mode assumption is appropriate provided the modes have a large frequency spacing relative to the detuning of the input field and the nonlinear frequency shifts.

Upon making the rotating-wave approximation and noting the Hamiltonian is defined to be normal ordered we obtain

$$H = \hbar\omega_C a^\dagger a + \hbar\chi'' a^{\dagger 2} a^2 \tag{2.5}$$

where the anharmonicity parameter is defined as

$$\chi'' = \left(\frac{3\hbar\omega^2}{8\epsilon_0^2}\right) \int \chi^{(3)}(\boldsymbol{r}) |u(\boldsymbol{r})|^4 \, d^3\boldsymbol{r}. \tag{2.6}$$

This quartic-mode integral for the anharmonicity parameter includes any effects due to spatial variation in the field intensity in the nonlinear medium. It can be verified that the use of a standing-wave mode increases the anharmonicity by a factor of 1·5 relative to the travelling-wave-mode case.

We point out here that the Hamiltonian we have derived is the anharmonic oscillator Hamiltonian in the rotating-wave approximation. Thus, although the effects we derive are novel for optical systems, bistable behaviour in the classical anharmonic oscillator is well known.

If we include the Hamiltonian for the coherent driving field and for a loss mechanism due to cavity damping we find for the total Hamiltonian

$$\begin{aligned} H &= \sum_{i=1}^{4} H_i \\ H_1 &= \hbar\omega_c a^\dagger a \\ H_2 &= \hbar\chi'' a^{\dagger 2} a^2 \\ H_3 &= i\hbar(a^\dagger E(t)\, e^{-i\omega_L t} - aE^*(t)\, e^{i\omega_L t}) \\ H_4 &= a^\dagger \Gamma_F + a\Gamma_F^\dagger. \end{aligned} \tag{2.7}$$

Here ω_C is the fundamental cavity resonance, χ'' is the anharmonicity, $E(t)$ is the driving field amplitude and ω_L the driving frequency while Γ_F, $\Gamma_F^\dagger$ are the reservoir operators for the cavity damping. This Hamiltonian is exact within the single-mode and rotating-wave approximations.

In a reference system rotating at a frequency ω_L the master equation for the density operator of the cavity field mode is obtained by utilising standard techniques described in Louisell (1973):

$$\begin{aligned} \dot\rho &= \sum_{i=1}^{4} \mathscr{L}_i[\rho] \\ \mathscr{L}_1[\rho] &= -i\Delta\omega[a^\dagger a, \rho] \\ \mathscr{L}_2[\rho] &= -i\chi''[a^{\dagger 2} a^2, \rho] \\ \mathscr{L}_3[\rho] &= (E(t)a^\dagger - E^*(t)a, \rho) \\ \mathscr{L}_4[\rho] &= \kappa'(2a\rho a^\dagger - \rho a^\dagger a - a^\dagger a\rho + 2n^{\text{th}}[[a, \rho], a^\dagger]). \end{aligned} \tag{2.8}$$

Here κ' is the energy relaxation rate, n^{th} is the thermal occupation number due to gaussian fluctuations in the thermal reservoir Γ_F and the detuning $\Delta\omega \equiv \omega_C - \omega_L$.

We shall consider the effect of quantum fluctuations in the next section. For the moment we neglect quantum fluctuations and consider equations for the mean field amplitudes $\alpha = \langle a \rangle$. In the semiclassical approximation (i.e. assuming that correlation functions factorise)

$$\frac{\partial}{\partial t}\begin{bmatrix}\alpha \\ \alpha^*\end{bmatrix} = \begin{bmatrix} E(t) - \alpha f(\alpha\alpha^*) \\ E^*(t) - \alpha^* f^*(\alpha\alpha^*)\end{bmatrix} \tag{2.9}$$

where $f(\alpha\alpha^*) = \kappa + 2\chi\alpha\alpha^*$ and we have defined the parameters $\kappa = \kappa' + \mathrm{i}\Delta\omega$ giving the linear absorption and dispersion and $\chi = \mathrm{i}\chi''$ giving the nonlinear dispersion.

In the presence of a two-photon absorber, equation (2.9) have the same form with $\chi = \chi' + \mathrm{i}\chi''$ where χ' gives the nonlinear absorption (Chaturvedi *et al* 1977).

If we define the mean photon number $n = \alpha\alpha^*$ the equations for the steady state following from equations (2.9) are

$$\begin{aligned} |E|^2 &= n|f(n)|^2 \\ &= n[\kappa'^2 + (\Delta\omega + 2n\chi'')^2]. \end{aligned} \tag{2.10}$$

We now investigate the regions of stability of this equation by a linearised analysis about the steady state.

Introducing small fluctuations about the steady state

$$\alpha(t) = \alpha_0 + \alpha_1(t) \tag{2.11}$$

we obtain the following linearised equations for the fluctuations:

$$\frac{\partial}{\partial t}\begin{bmatrix}\alpha_1(t) \\ \alpha_1^*(t)\end{bmatrix} = -\mathbf{A}\begin{bmatrix}\alpha_1(t) \\ \alpha_1^*(t)\end{bmatrix} \tag{2.12}$$

where

$$\mathbf{A} = \begin{bmatrix} n\dfrac{\partial f}{\partial n} + f(n), & \alpha_0^2\dfrac{\partial f}{\partial n} \\ \alpha_0^{*2}\dfrac{\partial f^*}{\partial n}, & n\dfrac{\partial f^*}{\partial n} + f^*(n)\end{bmatrix}. \tag{2.13}$$

Using the Hurwitz criterion for stability one finds that to obtain stable eigenvalues it is necessary to have

$$\mathrm{Tr}(\mathbf{A}) = 2\,\mathrm{Re}\left(f(n) + n\frac{\partial f}{\partial n}\right) > 0 \tag{2.14}$$

$$\mathrm{Det}(\mathbf{A}) = |f(n)|^2 + n\left(f^*(n)\frac{\partial f}{\partial n} + f(n)\frac{\partial f^*}{\partial n}\right) > 0. \tag{2.15}$$

A change in stability properties can occur if either $\mathrm{Tr}(A)$ or $\mathrm{Det}(A)$ changes sign. The points where $\mathrm{Det}(A)$ vanishes are soft-mode instabilities (since one of the eigenvalues is zero). In fact, $\mathrm{Det}(A) = 0$ is equivalent to finding a turning point in the state equation because

$$\mathrm{Det}(A) = \partial|E|^2/\partial n. \tag{2.16}$$

If $\mathrm{Det}(A)$ is non-vanishing but $\mathrm{Tr}(A)$ is zero there is a hard-mode instability with the onset of oscillations.

In the present situation of the nonlinear dispersive medium we find

$$\mathbf{A} = \begin{bmatrix} \kappa + 4\chi n, & 2\chi\alpha_0^2 \\ 2\chi^*\alpha_0^{*2}, & \kappa^* + 4\chi^* n\end{bmatrix} \tag{2.17}$$

with

$$\mathrm{Tr}(A) = 2\kappa'.$$

$$\mathrm{Det}(A) = 12|\chi|^2 n^2 + 4n[\chi\kappa^* + \chi^*\kappa] + |\kappa|^2.$$

For a linear loss mechanism $\kappa' > 0$ so that $\mathrm{Tr}(A) > 0$. (It is possible to get $\kappa' < 0$ in a linear amplifier, i.e. a laser with an external field resulting in an instability.)

The threshold points for dispersive optical bistability are $\mathrm{Det}(A) = 0$. This yields the state equation

$$|E|^2 = n[\kappa'^2 + (2\chi'' n + \Delta\omega)^2] \tag{2.18}$$

with turning points

$$n^{\pm} = [-2\Delta\omega \pm (\Delta\omega^2 - 3\kappa'^2)^{1/2}]/6\chi''. \tag{2.19}$$

It can be verified that for $n < n^-$ or $n > n^+$ the Hurwitz criterion gives stable eigenvalues, while for intermediate values of n there is an unstable branch. Hence provided $n^{\pm}$ are positive and real, the deterministic equations predict bistability with respect to small phase and amplitude fluctuations. It is clear that bistability requires that the detuning exceed a critical value, $\Delta\omega^2 > 3\kappa'^2$, and that the sign of the detuning is opposite to that of the anharmonicity, $\Delta\omega\chi'' < 0$. Hence for a positive χ'', to obtain bistability the zero-field cavity resonance ω_C must be at a lower frequency than the driving frequency ω_L. In physical terms, the intensity-dependent refractive index of the intracavity medium will increase the effective cavity resonance frequency to cause dispersive bistability as the input intensity is increased through the bistable region.

The state equation for bistability is displayed in figures 1 and 2 for different values of the detuning $\Delta\omega$, showing that the existence of bistability depends on the detuning.

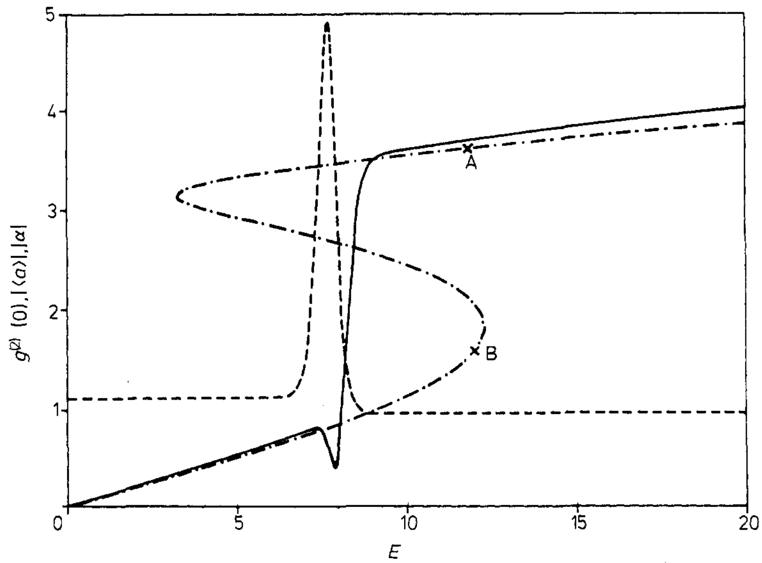

Figure 1. Chain curve, semiclassical value of steady-state field amplitude $|\alpha|$ as a function of driving field E; full curve, quantum-mechanical mean of steady-state field amplitude $|\langle a\rangle|$ as a function of driving field E; broken curve, second-order correlation function $g^{(2)}(0)$ as a function of driving field E. Detuning $\Delta\omega\chi'' < 0$ (parameters $\Delta\omega = -10$, $\kappa' = 1$, $\chi'' = 0{\cdot}5$).

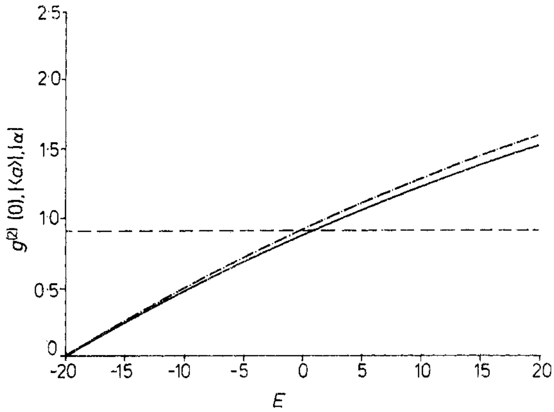

Figure 2. As for figure 1, but with detuning $\Delta\omega\chi'' > 0$ (parameters $\Delta\omega = 10, \kappa' = 1, \chi'' = 0{\cdot}5$).

3. Quantum fluctuations via the Fokker–Planck equation

The Fokker–Planck equation corresponding to the master equation can now be obtained via standard methods using the Glauber representation (Glauber 1963a, b, Louisell 1973). However, the resulting Fokker–Planck equation does not always have solutions except as generalised functions: that is, the diagonal P representation does not always exist†. For this reason we prefer to use the non-diagonal generalised P representation defined by

$$\rho = \int_{\mathscr{D}} P(\alpha, \beta)\left(\frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle}\right) \mathrm{d}\mu(\alpha, \beta) \tag{3.1}$$

as this always gives solutions on an appropriate domain (Drummond 1979, Drummond and Gardiner 1979). Here $\mathscr{D}$ is the integration domain and $\mathrm{d}\mu$ is the integration measure. In §§ 3 and 4 the integration measure is the volume integral $\mathrm{d}^2\alpha\ \mathrm{d}^2\beta$ over a complex phase space; in § 5 the measure is a line-integral measure $\mathrm{d}\alpha\ \mathrm{d}\beta$ over a manifold embedded in a complex phase space. For later use, we write $(\alpha, \beta) = (\alpha, \alpha^\dagger)$ where $(\alpha, \alpha^\dagger)$ are not complex conjugate. However, there is the following correspondence principle between operators and c numbers:

$$\alpha \leftrightarrow a \qquad \alpha^\dagger \leftrightarrow a^\dagger.$$

The Fokker–Planck equation is as follows:

$$\frac{\partial}{\partial t} P(\boldsymbol{\alpha}) = \left(\frac{\partial}{\partial\alpha}(\kappa\alpha + 2\chi\alpha^2\alpha^\dagger - E(t)) - \chi\frac{\partial^2}{\partial\alpha^2}\cdot\alpha^2 \right.$$
$$+ \frac{\partial}{\partial\alpha^\dagger}(\kappa^*\alpha^\dagger + 2\chi^*\alpha^{\dagger 2}\alpha - E^*(t)) - \chi^*\frac{\partial^2}{\alpha^{\dagger 2}}\cdot\alpha^{\dagger 2}$$
$$\left. + \Gamma_1\frac{\partial^2}{\partial\alpha\,\partial\alpha^\dagger}\right) P(\boldsymbol{\alpha}) \tag{3.2}$$

† The (anti-normally ordered) Q representation $\langle\alpha|\rho|\alpha\rangle$, despite its existence and positive definiteness being guaranteed, is also not useful since the resulting Fokker–Planck equation has a non-positive definite diffusion matrix and exact solutions as in § 5 are not obtained.

where we have defined $\Gamma_1 = 2\kappa' n^{\text{th}}$. Here $\boldsymbol{\alpha} = (\alpha, \alpha^\dagger)$ is a vector (in two-dimensional complex space), the argument of the generalised P function.

We now turn to the derivation of stochastic differential equations from the Fokker–Planck equation. When the normal (diagonal) P function representation is used with $\alpha^* = \alpha^\dagger$ it is readily shown that the presence of nonlinear terms means that the Fokker–Planck diffusion is non-positive definite. This means that the usual Ito theorems for stochastic differential equations are not applicable (Arnold 1974). However, as shown by Drummond and Gardiner (1979), the Fokker–Planck equation in $(\alpha, \alpha^\dagger)$ can be transformed to a four-dimensional equation with positive definite diffusion.

The exact stochastic differential equations in the Ito calculus are obtained on transforming the Fokker–Planck equation into the Ito form:

$$\frac{\partial}{\partial t}\begin{bmatrix}\alpha\\ \alpha^+\end{bmatrix} = \begin{bmatrix}E(t) - \kappa\alpha - 2\chi\alpha^2\alpha^\dagger\\ E^*(t) - \kappa^*\alpha^\dagger - 2\chi^*\alpha^{\dagger 2}\alpha\end{bmatrix} + \begin{bmatrix}-2\chi\alpha^2, & \Gamma_1\\ \Gamma_1, & -2\chi^*\alpha^{\dagger 2}\end{bmatrix}^{1/2}\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}. \tag{3.3}$$

Here ξ_1, ξ_2 are delta-correlated random gaussian functions so that $\alpha, \alpha^\dagger$ are complex conjugate in the mean.

We may allow for fluctuations in the driving field by considering a simple model of coherent driving field plus thermal fluctuations:

$$E(t) = E_0 + \delta E(t) \tag{3.4}$$

where

$$\langle \delta E^*(t)\delta E(t)\rangle = \Gamma_E \delta(t - t').$$

This can be treated very simply within the framework of equations (3.3) by including this additional fluctuation term with the fluctuations already present to give a total non-diagonal term of

$$\Gamma = \Gamma_1 + \Gamma_E. \tag{3.5}$$

Thus the overall stochastic differential equation would be

$$\frac{\partial}{\partial t}\begin{bmatrix}\alpha^\dagger\\ \alpha\end{bmatrix} = \begin{bmatrix}E_0 - \kappa\alpha - 2\chi\alpha^2\alpha^\dagger\\ E_0^* - \kappa^*\alpha^\dagger - 2\chi^*\alpha^{\dagger 2}\alpha\end{bmatrix} + \begin{bmatrix}-2\chi\alpha^2, & \Gamma\\ \Gamma, & -2\chi^*\alpha^{\dagger 2}\end{bmatrix}^{1/2}\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}. \tag{3.6}$$

We will utilise these equations to obtain the deterministic state equations and the linearised response to fluctuations. It should be noted that the generalised P representation as defined in equation (3.1) is a normally ordered representation and hence normally ordered averages are obtained via

$$\langle (a^+)^n (a)^m\rangle = \int P(\alpha, \beta)\beta^n\alpha^m \, d\mu(\alpha, \beta). \tag{3.7}$$

4. Linearised fluctuation theory

We now proceed to analyse the behaviour of the stochastic differential equations (3.6) close to a stable branch. We adopt an asymptotic expansion valid for small fluctuations (this may be formally shown to be an expansion in σ, where σ is the fluctuation variance

(Gardiner and Chaturvedi 1977)). To first order in the fluctuations, $\boldsymbol{\alpha}_1(t)$ obeys the equation

$$\frac{\partial}{\partial t}\boldsymbol{\alpha}_1(t) = -\mathbf{A}\,.\,\boldsymbol{\alpha}_1(t) + \mathbf{D}^{1/2}(\boldsymbol{\alpha}_0)\,.\,\boldsymbol{\xi}(t). \tag{4.1}$$

Here $\mathbf{A}$ is the linearised drift and $\mathbf{D}$ is the diffusion array evaluated at $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$. These have the general form

$$\mathbf{D} = \begin{bmatrix} -2\chi\alpha_0^2, & \Gamma \\ \Gamma, & -2\chi^*\alpha_0^{*2} \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} \kappa + 4\chi n, & 2\chi\alpha_0^2 \\ 2\chi^*\alpha_0^{*2}, & \kappa^* + 4\chi^* n \end{bmatrix}. \tag{4.2}$$

The correlation matrix

$$\mathbf{C} = \begin{bmatrix} \langle a^2\rangle - \langle a\rangle^2, & \langle a^\dagger a\rangle - |\langle a\rangle|^2 \\ \langle a^\dagger a\rangle - |\langle a\rangle|^2, & \langle a^{\dagger 2}\rangle - \langle a^\dagger\rangle^2 \end{bmatrix} \tag{4.3}$$

$$= \begin{bmatrix} \langle \alpha_1^2\rangle, & \langle \alpha_1^\dagger \alpha_1\rangle \\ \langle \alpha_1^\dagger \alpha_1\rangle, & \langle \alpha_1^{\dagger 2}\rangle \end{bmatrix}$$

can be evaluated using the method of Chaturvedi *et al* (1977b):

$$\mathbf{C} = \left(\frac{\mathbf{D}\,.\,\mathrm{Det}(\mathbf{A}) + (\mathbf{A} - \mathrm{I}\,.\,\mathrm{Tr}(\mathbf{A}))\mathbf{D}(\mathbf{A} - \mathrm{I}\,.\,\mathrm{Tr}(\mathbf{A}))^{\mathrm{T}}}{2\,\mathrm{Tr}(\mathbf{A})\,\mathrm{Det}(\mathbf{A})}\right)$$

$$= \left(\frac{1}{2\kappa'\lambda}\right)\begin{bmatrix} -2\chi\alpha_0^2(\kappa + 4\chi n)^*(\Gamma + \kappa'), & \Gamma|\kappa + 4\chi n|^2 + 4\chi''^2 n^2\kappa' \\ \Gamma|\kappa + 4\chi n|^2 + 4\chi''^2 n^2 \kappa', & -2\chi^*\alpha_0^{*2}(\kappa + 4\chi n)(\Gamma + \kappa') \end{bmatrix} \tag{4.4}$$

where

$$\lambda = \kappa'^2 + \Delta\omega^2 + 8\Delta\omega\,.\,\chi'' n + 12\chi''^2 n^2.$$

4.1. *Transmitted intensity*

The total intensity in the cavity (or total photon number) is, to first order in the asymptotic expansion,

$$\bar{n} = |\alpha_0|^2 + \langle \alpha_1^\dagger \alpha_1\rangle$$

$$= n + \left(\frac{1}{2\kappa'\lambda}\right)(\Gamma|\kappa + 4\chi n|^2 + 4\chi''^2 n^2 \kappa'). \tag{4.5}$$

Thus there is a coherent and an incoherent part to the total intensity. We first look at the linear cavity limit where $\chi'' \to 0$. In this limit the incoherent intensity is due to thermal fluctuations. Provided the driving field is coherent we obtain

$$\bar{n} = n + n^{\mathrm{th}}, \tag{4.6}$$

that is, the intensities of the coherent field and the thermal background are additive in a linear cavity. This is to be expected as they are uncorrelated. In the nonlinear situation we obtain

$$\bar{n} = n + n^{\mathrm{th}}\left(\frac{|\kappa + 4\chi n|^2}{\lambda}\right) + 2\chi''^2 n^2/\lambda. \tag{4.7}$$

Thus the analysis would predict an increase in the background fluctuation due to the nonlinearity with a maximum at $\lambda = 0$. This divergence occurs at the instability points

$n^{\pm}$, which are the points where the linear theory breaks down. Finally, there is an additional term proportional to $n^2\chi''^2$. This gives the intensity of quantum fluctuations in the system and is due to the nonlinearity in the polarisability.

4.2. Second-order correlation function

The effect of both quantum fluctuations and thermal fluctuations is to increase the total photon number. However, information on the photon statistics given by the second-order correlation function shows that these are physically different types of process. The second-order correlation function is defined by (Glauber 1963a, b)

$$g^{(2)}(\tau)=\frac{\langle E^-(t)E^-(t+\tau)E^+(t+\tau)E^+(t)\rangle}{\langle E^-(t)E^+(t)\rangle\langle E^-(t+\tau)E^+(t+\tau)\rangle}. \tag{4.8}$$

We shall consider the one-time correlation function $g^{(2)}(0)$. While thermal fluctuations always increase $g^{(2)}(0)$ above the input value of unity, for a coherent driving field the quantum noise term can decrease $g^{(2)}(0)$ to a value less than one, indicative of photon antibunching statistics. Photon antibunching—a property predicted by quantum electrodynamics—has recently been observed (Kimble *et al* 1977) in resonance fluorescence from a two-level atom in agreement with theoretical predictions (Carmichael and Walls 1976a, b, Kimble and Mandel 1976, Cohen Tannoudji 1977).

We shall calculate the correlation function $g^{(2)}(0)$ to first order in the asymptotic expansion

$$\begin{aligned} g^{(2)}(0) &= 1+\frac{2}{n}[\langle\alpha_1^+\alpha_1\rangle+\mathrm{Re}(\alpha_0^*\langle\alpha_1^2\rangle/\alpha_0)] \\ &= 1+[(2n^{\mathrm{th}}|k+4\chi n|^2+4\chi''^2n^2)/(\lambda n)-2\chi''(\Delta\omega+4\chi''n)(1+2n^{\mathrm{th}})/\lambda]. \end{aligned} \tag{4.9}$$

Here the coefficient of n^{th} is positive definite while the term arising from quantum fluctuations may be negative, thereby giving rise to photon antibunching for coherent driving fields. The effect in the limit of $n\to\infty$ is not dependent on detuning, and is given by

$$g^{(2)}(0)=1-\tfrac{1}{3}n^{-1}. \tag{4.10}$$

For lower n values, in the bistable region, this antibunching disappears and is replaced by photon bunching. However, by looking at the correlation function in the case of detuning in the opposite direction to the bistable case (i.e. by allowing $\Delta\omega\chi''>0$), it can be shown that with a coherent driving field and zero-temperature reservoirs, there is photon antibunching for all input fields. In principle this offers the possibility of a direct measurement of photon antibunching without the additional atomic number fluctuations (Jakeman *et al* 1977, Carmichael *et al* 1978, Kimble *et al* 1978) which entered the experimental observation of Kimble *et al* (1977) in resonance fluorescence. In this respect the effect is similar to several other suggested nonlinear optical experiments for the observation of photon antibunching (Stoler 1974, Simaan and Loudon 1975, Chaturvedi *et al* 1977, Mostowski and Rzazewski 1978, Drummond *et al* 1979).

4.3. Spectrum of the transmitted light

The spectrum may be obtained directly following the method of Chaturvedi *et al*

(1977b), which gives

$$S(\omega_L+\omega)=n\delta(\omega)+\left(\frac{1}{2\pi}(\mathbf{A}+I.\,\mathrm{i}\omega)^{-1}\mathbf{D}(\mathbf{A}^{\mathrm{T}}-I.\,\mathrm{i}\omega)^{-1}\right)_{21}$$

$$=n\delta(\omega)+\left(\frac{1}{2\pi|\lambda(\omega)|^2}\right)[8\kappa'\chi''^2n^2+\Gamma(|\kappa+4\chi n+\mathrm{i}\omega|^2+4\chi''^2n^2)] \tag{4.11}$$

where

$$\lambda(\omega)\equiv(\mathrm{i}\omega+\kappa^*+4\chi^*n)(\mathrm{i}\omega+\kappa+4\chi n)-4|\chi|^2n^2 .$$

This spectrum contains a delta-function peak corresponding to the radiation transmitted at the input frequency plus two quasi-Lorentzian peaks located symmetrically about the input frequency at frequencies

$$\omega^{\pm}=\omega_L\pm(\Delta\omega^2+8\Delta\omega\chi''n+12n^2\chi''^2)^{1/2} . \tag{4.12}$$

These lines coalesce to give a single-peaked spectrum when

$$\Delta\omega^2+8\Delta\omega\chi''n+12n^2\chi''^2\leq 0. \tag{4.13}$$

In order to understand the effect of the nonlinearity on the spectrum we first compare this result with the case of a linear interferometer.

The spectrum in this case is obtained by setting $\chi''=0$ in equation (4.11):

$$S(\omega_L+\omega)=n\delta(\omega)+\left(\frac{1}{2\pi|\lambda(\omega)|^2}\right)(\Gamma|\kappa+\mathrm{i}\omega|^2). \tag{4.14}$$

In this case we see that the pole at $\mathrm{i}\omega=-\kappa$ vanishes, leaving a Lorentzian peak at $\omega=\omega_C-\omega_L$. Thus the final form is (Louisell 1973)

$$S(\omega)=\frac{|E|^2\delta(\omega-\omega_L)}{(\kappa'^2+\Delta\omega^2}+\left(\frac{1}{2\pi}\right)\left(\frac{\Gamma}{\kappa'^2+(\omega-\omega_c)^2}\right). \tag{4.15}$$

Thus the overall spectrum is non-symmetric for $\omega_C\neq\omega_L$. There is an 'elastic' peak at the driving frequency ω_L and a Lorentzian peak at the interferometer tuning ω_C due to thermal fluctuations. If we compare this result with the nonlinear interferometer, it approaches the linear case at low enough driving fields.

However as $|E|$ is increased, the tuning point varies since the cavity refractive index is power-dependent when χ'' is non-zero. In addition, both symmetrically placed peaks can develop a finite peak height. The spectrum is plotted in figures 3 and 4 for various values of the driving field $|E|$. In the absence of thermal fluctuations ($\Gamma=0$), the spectrum arises entirely from the quantum fluctuations and is completely symmetric relative to ω_L. In general the spectrum is double-peaked, although there is a certain range of photon numbers where the lines coalesce to give a single line. This occurs at

$$\tfrac{1}{6}|\Delta\omega|\leq|\chi''|n\leq\tfrac{1}{2}|\Delta\omega|. \tag{4.16}$$

As the threshold point is approached from below, the initially separated spectral lines coalesce, then one line diverges at the threshold point. Above the threshold a double-peaked spectrum appears on the upper branch. On reducing the driving intensity, the lines coalesce then diverge at the lower threshold.

When thermal fluctuations are included the line closest to the effective interferometer tuning is enhanced at low driving intensity. This corresponds to the linear result. However, when the input intensity is increased the spectral lines coalesce and

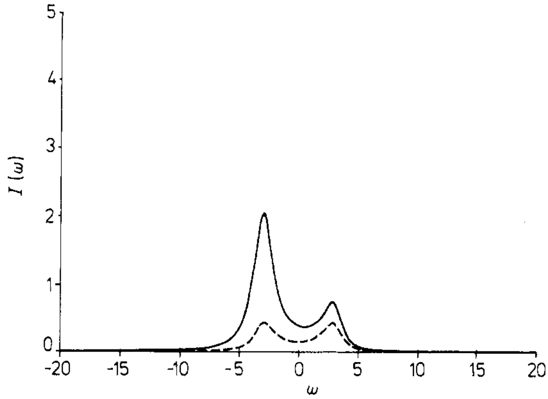

Figure 3. Spectrum of the transmitted light below threshold (point B on figure 1). Broken curve, quantum fluctuations only; full curve, quantum plus thermal fluctuations.

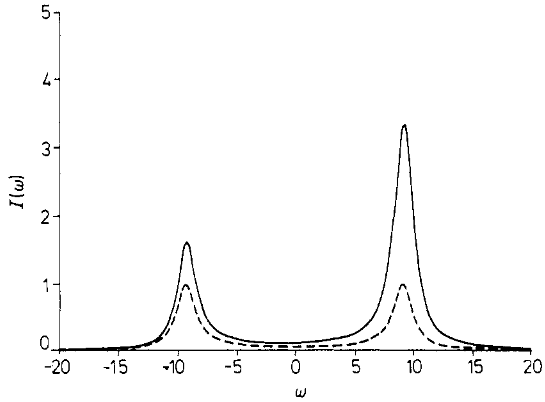

Figure 4. Spectrum of the transmitted light above threshold (point A on Figure 1). Broken curve quantum fluctuations only; full curve, quantum plus thermal fluctuations.

then a crossing-over effect occurs. Above the bistable threshold the line furthest away from the original linear tuning becomes relatively enhanced. This occurs because the effective refractive index of the medium is changed by the driving field.

A physical understanding of the spectrum may be obtained by considering the underlying quantum process†. In the case of a perfectly coherent driving field we consider the incident scattering process involving two laser photons giving rise to the appearance of two sidebands. This leads to a two-peaked spectrum symmetrically placed about ω_L and with equal peak heights since photons at frequency ω_c' and $2\omega_L - \omega_c'$ are produced in the same scattering process (figure 5(a)).

The asymmetric spectrum in the case of thermal fluctuations may be understood by considering the process depicted in figure 5(b), which occurs in addition to the process above. The emission at the cavity frequency is enhanced either by stimulated emission of a thermal photon ($\omega_L > \omega_c'$) or by absorption of a thermal photon ($\omega_L < \omega_c'$), which enables energy to be conserved. Thus when thermal radiation is present emission at the cavity frequency is enhanced relative to the other emission peak at $2\omega_L - \omega_c'$. It should

† We wish to thank C Cohen Tannoudji for discussions on this point.

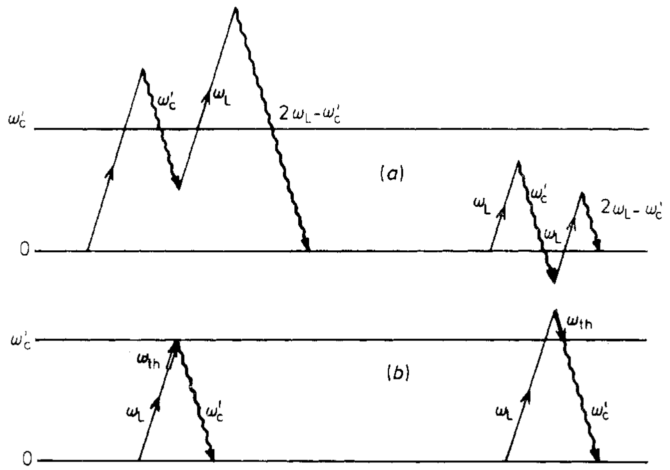

Figure 5. (*a*) Photon scattering process with coherent light. (*b*) Photon scattering process when thermal fluctuations are present.

be noted that in figure 5 ω_c' refers to the effective cavity frequency including changes to the nonlinear refractive index.

A similar behaviour is observed in the case of detuned atomic fluorescence with a fluctuating driving field (Kimble and Mandel 1977, Wodkiewicz 1978, Knight *et al* 1978, Hassan and Bullough 1979). In this case there is a symmetric spectrum with a coherent input that becomes asymmetric when fluctuations are included in the input.

5. Exact photon statistics

In the previous sections the effect of quantum fluctuations was calculated to first order in the fluctuation variance. It is, however, of interest to calculate moments in regions where the linearisation procedure is not valid. This may be achieved in the steady state by means of an exact solution of the Fokker–Planck equation which exists whenever the 'potential' equations (Haken 1975) are satisfied†.

Consider the Fokker–Planck equation

$$\frac{\partial}{\partial t} P(\boldsymbol{\alpha}) = (\partial_\mu A_\mu(\boldsymbol{\alpha}) + \tfrac{1}{2}\partial_\mu \partial_\nu D_{\mu\nu}(\boldsymbol{\alpha}))P(\boldsymbol{\alpha}). \tag{5.1}$$

A steady-state solution exists provided the potential conditions

$$\partial_\mu V_\nu = \partial_\nu V_\mu \tag{5.2}$$

are satisfied, where

$$V_\rho(\boldsymbol{\alpha}) = (D_{\rho\nu}(\boldsymbol{\alpha}))^{-1}(2A_\nu(\boldsymbol{\alpha}) + \partial_\sigma D_{\nu\sigma}(\boldsymbol{\alpha})). \tag{5.3}$$

These conditions are not satisfied in general for the Fokker–Planck equation (3.2). One may consider the limits $\Gamma \gg |\chi n|$ or $\Gamma \ll |\chi n|$. These are the 'thermal' or 'quantum' noise limits, respectively.

† Although potential solutions have previously only been used in cases of thermal noise, a similar mathematical procedure produces a solution to the present Fokker–Planck equation in the quantum noise limit.

For dispersive optical bistability no potential solution exists in the thermal limit. This is in constrast to absorptive optical bistability where a potential solution has been obtained in the thermal limit (Bonifacio and Lugiato 1978, Schenzle and Brandt 1978). In the quantum limit, that is for a perfectly coherent driving field and $kT \ll \hbar\omega_L$, we can neglect thermal fluctuations, that is set $\Gamma = 0$. The diffusion array is then

$$\mathbf{D} = \begin{bmatrix} -2\chi\alpha^2 & \\ & -2\chi^*\alpha^{\dagger 2} \end{bmatrix} \tag{5.4}$$

$$\mathbf{A} = \begin{bmatrix} \kappa\alpha + 2\chi\alpha^2\alpha^\dagger - E_0 \\ \kappa^*\alpha^\dagger + 2\chi^*\alpha^{\dagger 2}\alpha - E_0^* \end{bmatrix}. \tag{5.5}$$

The calculation of $\boldsymbol{V}$ is straightforward:

$$\begin{aligned} V_1 &= -\left(\frac{1}{\chi}\right)\left(\frac{\bar{\kappa}}{\alpha} + 2\chi\alpha^\dagger - E_0/\alpha^2\right) \\ V_2 &= -\left(\frac{1}{\chi^*}\right)\left(\frac{\bar{\kappa}^*}{\alpha^\dagger} + 2\chi^*\alpha - E_0^*/\alpha^{\dagger 2}\right) \end{aligned} \tag{5.6}$$

where we have defined $\bar{\kappa} = \kappa - 2\chi$. This yields

$$\partial_1 V_2 = \partial_2 V_1 = -2 \tag{5.7}$$

and the equality of cross derivations is thus obtained directly.

The next step is to integrate the generalised force $\boldsymbol{V}$ to obtain the potential function and steady-state distribution:

$$\begin{aligned} P(\boldsymbol{\alpha})_{SS} &= \exp\left(-\int^{\boldsymbol{\alpha}} V_\rho(\boldsymbol{\alpha}')\,\mathrm{d}\alpha'_\rho\right) \\ &= \exp\left\{\int^{\boldsymbol{\alpha}}\left[\frac{1}{\chi}\left(\frac{\bar{\kappa}}{\alpha_1} + 2\chi\alpha_1^\dagger - \frac{E_0}{\alpha_1^2}\right)\mathrm{d}\alpha_1 \right.\right. \\ &\quad \left.\left. + \frac{1}{\chi^*}\left(\frac{\bar{\kappa}^*}{\alpha_1^\dagger} + 2\chi^*\alpha_1 - \frac{E_0^*}{\alpha_1^{\dagger 2}}\right)\mathrm{d}\alpha_1^\dagger\right]\right\} \\ &= \exp[(\bar{\kappa}/\chi)\ln\alpha + (\bar{\kappa}/\chi)^*\ln\alpha^\dagger + (E_0/\chi\alpha) + (E_0^*/\chi^*\alpha^\dagger) + 2\alpha\alpha^\dagger]. \end{aligned} \tag{5.8}$$

We may simplify this by defining the driving phase so that (E_0/χ) is real, which gives

$$P(\boldsymbol{\alpha}) = \alpha^{(c-2)}\alpha^{\dagger(d-2)}\exp\left[\left(\frac{E_0}{\chi}\right)\left(\frac{1}{\alpha} + \frac{1}{\alpha^\dagger}\right) + 2\alpha\alpha^\dagger\right] \tag{5.9}$$

where

$$c = (\kappa/\chi) \qquad d = (\kappa/\chi)^*.$$

It can be seen immediately that the usual integration domain of the complex plane with $\alpha^\dagger = \alpha^*$ is not possible since the potential diverges for $\alpha\alpha^\dagger \to \infty$. This means that no Glauber–Sudarshan P function exists in the steady state (except as a generalised function). Instead it is necessary to choose a generalised P representation, which corresponds to an expansion of ρ with non-diagonal coherent state projection operators. This implies an integration domain with $\alpha^\dagger \neq \alpha^*$, defined so the distribution function vanishes correctly at the boundary. This means choosing new paths of integration for α, $\alpha^\dagger$ which are to be line integrals on the individual $(\alpha, \alpha^\dagger)$ complex planes. In other words, the new domain will be a complex manifold embedded in the space $\mathbb{C}^2$. Firstly a variable change is made to $\beta = 1/\alpha$, $\beta^\dagger = 1/\alpha^\dagger$. The normalisation integral is then obtained (where $\mathscr{C}$ is the integration path):

$$I(c, d) = \iint_{\mathscr{C}} \sum_{n=0}^{\infty} \left[\frac{2^n}{n!}\right] \beta^{-c-n} \beta^{\dagger -d-n} \exp\left(\frac{E_0}{\chi}(\beta + \beta^\dagger)\right) \mathrm{d}\beta \, \mathrm{d}\beta^\dagger. \tag{5.10}$$

The integrand is now in a recognisable form as corresponding to a sum of gamma function integrals. It is therefore appropriate to define each path of integration to be a Hankel path of integration, from $(-\infty)$ on the real axis around the origin in an anticlockwise direction and back to $(-\infty)$. With this definition of the integration domain, the following gamma-function identity holds (Abramowitz and Stegun 1964):

$$(\Gamma(c+n))^{-1} = \left(\frac{t^{1-c-n}}{2\pi \mathrm{i}}\right) \int_{\mathscr{C}} \beta^{-c-n} \exp(\beta t) \, \mathrm{d}\beta. \tag{5.11}$$

Hence, applying this result to both β and $\beta^\dagger$ integrations, one obtains

$$I(c, d) = -4\pi^2 \sum_{n=0}^{\infty} \left(\frac{2^n (E_0/\chi)^{c+d+2(n-1)}}{n!\,\Gamma(c+n)\Gamma(d+n)}\right). \tag{5.12}$$

The series is a transcendental function which can be written in terms of the generalised Gauss hypergeometric series. That is, there is a hypergeometric series called ${}_0F_2$ which is defined as (Gradshteyn and Ryzhik 1965)

$${}_0F_2(c, d, z) = \sum_{n=0}^{\infty} \left(\frac{z^n \Gamma(c)\Gamma(d)}{\Gamma(c+n)\Gamma(d+n) n!}\right). \tag{5.13}$$

From now on, for simplicity, we will write just $F(\;\;)$, instead of ${}_0F_2(\;\;)$. Now the normalisation integral can therefore be rewritten in the form

$$I(c, d) = \left(\frac{-4\pi^2 |E_0/\chi|^{c+d-2}}{\Gamma(c)\Gamma(d)}\right) F(c, d, 2|E_0/\chi|^2). \tag{5.14}$$

The moments of the distribution function divided by the normalisation factor give all the observable one-time correlation functions. Luckily the moments have exactly the same functional form as the normalisation factor (with the replacement of (c, d) by $(c+i, d+j)$) so that no new integrals need to be calculated. The ith-order correlation function is

$$G^{(i)} = \langle (a^\dagger)^i (a)^i \rangle_\rho \tag{5.15}$$

$$= \left(\frac{|E_0/\chi|^{2i} \Gamma(c)\Gamma(d) F(i+c, i+d, 2|E_0/\chi|^2)}{\Gamma(i+c)\Gamma(i+d) F(c, d, 2|E_0/\chi|^2)}\right). \tag{5.16}$$

This is the general expression for the ith order correlation function of a nonlinear dispersive cavity with a coherent driving field and zero-temperature heat baths.

The results for the mean amplitude $\langle a \rangle$ and correlation function $g^2(0)$ are

$$\langle a \rangle = \frac{1}{c} |E_0/\chi| F(1+c, d, 2|E_0/\chi|^2) / F(c, d, 2|E_0|\chi|^2) \tag{5.17}$$

$$g^{(2)}(0) = \left(\frac{cdF(c, d, 2|E_0/\chi|^2) F(c+2, d+2, 2|E_0/\chi|^2)}{(c+1)(d+1)[F(c+1, d+1, 2|E_0/\chi|^2)]^2}\right). \tag{5.18}$$

These quantities are plotted in figures 1 and 2, where they are compared with the semiclassical value for $\langle a \rangle$. It is seen that, whereas the semiclassical equation predicts a bistability or hysteresis, the exact steady-state equation which includes quantum fluctuations does not exhibit bistability or hysteresis.

The extent to which bistability is observed in practice will depend on the fluctuations, which in turn determine the time for random switching from one branch to the other. The driving field must be ramped in time intervals shorter than this random switching time in order for bistability to be observed.

The variance of the fluctuations as displayed by $g^{(2)}(0)$ show an increase as the fluctuations are enhanced near the transition point. The dip in the steady-state mean at the transition point is due to out-of-phase fluctuations between the upper and lower branches.

We note that the exact steady-state mean $|\bar{\alpha}|$ does not cut the semiclassical curve in two equal areas, i.e. it does not give a Maxwell construction. This is an important difference of nonequilibrium phase transitions compared with equilibrium phase transitions. Mathematically, it is a consequence of the nonconstant coefficient of the diffusion term in the Fokker–Planck equation. This serves to indicate the essentially different nature of the fluctuations. In this case the fluctuations are quantum in origin, though similar effects occur in other non-equilibrium systems, e.g. chemical instabilities (Matheson *et al* 1975, Janssen 1974).

6. Summary

A quantum treatment of a coherently driven nonlinear dispersive Fabry–Perot cavity has been given. The semiclassical equations predict bistability and hysteresis. The spectrum of the transmitted light which exhibits a single peak near the threshold of the lower branch splits into two peaks as the system makes the transition to the upper branch. For a coherent driving field these peaks are symmetrically placed about the frequency of the driving field. If fluctuations are present in the driving field the spectrum becomes asymmetric as the peak at the interferometer resonance is enhanced. The photon statistics of the transmitted light are studied through a calculation of $g^{(2)}(0)$. It is shown that fluctuations in the driving field are reduced on the upper branch. For a coherent driving field the transmitted light on the upper branch will exhibit the property of photon antibunching. This effect is enhanced with detuning in the opposite direction to that for bistability.

In the absence of thermal fluctuations an exact expression for the photon distribution function in the steady state was obtained. The distribution obtained corresponds to a density matrix with a non-diagonal representation in terms of coherent states (generalised P representation). The generalised P representation allows for an analytic expression to be obtained for the distribution function for fields which exhibit photon antibunching—a region where solutions for the Glauber–Sudarshan P representation are not possible except in terms of highly singular distributions. This steady-state distribution has a completely different character from the Landau–Ginsberg type of distribution associated with thermal fluctuations in equilibrium systems. In this case the fluctuations are quantum and the non-equilibrium nature of the transition is reflected by the absence of a Maxwell construction. The system studied, dispersive optical bistability, has great potential for use in optical communication systems. In addition, this system offers several possible interesting experiments involving photon statistics and spectral measurements to investigate the nature of quantum fluctuations in non-equilibrium systems.

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